ON MACROSCOPIC MEASURES OF PLASTIC WORK AND DEFORMATION IN MICRO-HETEROGENEOUS MEDIA

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Summary. Macroscopic properties of any elastoplastic medium with a heterogeneous micro-structure are considered. Results of a rigorous general theory due to Mandel and Hill are argumented and re-derived in a systematic manner. Ilyushin's postulate concerning the work expended in cycles of strain is also discussed from this standpoint.

1. Preliminaries. A rigorous general analysis of the macroscopic properties of elastoplastic continua which are heterogeneous on a micro-scale, such as composites or polycrystals, has been initiated in recent years by Mandel [1], Hill [2] and Rice [3]. Though differing markedly in scope and emphasis, the respective treatments have in common one distinctive feature: full account is taken of the heterogeneity of the microscopic fields of stress and strain, without idealizations or assumptions (except for the customary approximations of small-deformation theory). Some further results obtained by this approach are presented here. They concern mainly the connexions and distinctions between the definitions of plastic strain-rate energy dissipations at the macroscopic and microscopic levels.

We adopt Hill's specification of a representative volume element for a macroscopically homogeneous continuum. The virtual work principle applied to such an element provides a basic mean-value theorem which is used repeatedly in the general analysis. It states that the global volume average of the scalar product of any pair of stress and strain tensors is equal to the product of their separate averages, provided that the classical equations of stress equilibrium and strain compatibility are respectively satisfied throughout the representative element and provided also that the associated tractions and displacements over its surface are macroscopically uniform. This theorem can be written concisely as

$$\{\sigma\varepsilon\} = \{\sigma\}\ \{\varepsilon\} \tag{1}$$

where $\{\cdot\cdot\}$ signifies a global average and σ and ε are symbolic notation for local stress and infinitesimal strain. Similar equalities hold, of course, when either tensor is replaced by its differential or its rate of change (translational and rotational effects within the element being disregarded). The precise statement and proof of (1) are due to Hill [2, 4].

The plastic part of a local strain is often defined fromally to be

$$\varepsilon - M\sigma \tag{2}$$

where M is the (possibly anisotropic) fourth order tensor of elastic compliances in the current state. However, (2) only has a practical significance for a material that unloads elastically, without renewed plastic flow, during complete removal of the stress. For this and other reasons it is more fruitful to consider the plastic part of a differential increment of strain:

$$d\varepsilon - Md\sigma \tag{3}$$

In a metal that deforms plastically by crystallographic slip M remains constant so long

as the lattice geometry is undisturbed. But when M varies, as happens in some non-metals,

$$d\varepsilon - Md\sigma = d(\varepsilon - M\sigma) + (dM)\sigma \tag{4}$$

and then (2) and (3) must be distinguished.

According to the usual constitutive law for an elastoplastic solid the left-hand quantity in (4) is assumed to be directed along the outward normal to the yield surface in stress space. As was pointed out by Ilyushin [5], and proved in a wider context by Hill [6], such a "normality rule" is implied not only by Drucker's arbitrary definition of material stability but by Ilyushin's weaker postulate that positive work is expended in any closed cycle of strain. By contrast, as (4) makes plain, normality of $d(\varepsilon - M\sigma)$ is not implied when M varies; see Ilyushin [7] for extended discussion.

Along with (2) and (3) we shall subsequently consider their duals:

$$\sigma - L\varepsilon \tag{5}$$

and

$$d\sigma - Ld\varepsilon \tag{6}$$

where L is the fourth-order tensor of elastic moduli. Evidently (6) can be interpreted as the stress deficit after an infinitesimal cycle of strain in which an initial plastic flow is followed by elastic recovery. Corresponding to (4) we now have

$$d\sigma - Ld\varepsilon = d(\sigma - L\varepsilon) + (dL)\varepsilon \tag{7}$$

Since L and M are inverses an alternative expression for the right-hand side is

$$-Ld\,(\varepsilon - M\sigma) + (dL)M\sigma$$

which was used by Ilyushin [5] to discuss normality with regard to the yield surface in strain space.

It is convenient to note here the identity

$$\sigma(\varepsilon - M\sigma) + \varepsilon\,(\sigma - L\varepsilon) \equiv (\sigma - L\varepsilon)(\varepsilon - M\sigma) \tag{8}$$

together with similar ones in the differentials of stress and strain or in their rates of change. The right-hand side of (8) can obviously be written as a quadratic form with matrix L or M respectively:

$$-(\sigma - L\varepsilon)M(\sigma - L\varepsilon), \quad \text{or} \quad -(\varepsilon - M\sigma)L(\varepsilon - M\sigma)$$

The quantity (8) is therefore always negative for actual materials (or trivially zero when the deformation is purely elastic).

2. Macroscopic tensors. Each macroscopic tensor variable will be distinguished from its microscopic counterpart by using anglular brackets $\langle\ \rangle$. By definition the macroscopic stress and strain are just their volume averages:

$$\langle\sigma\rangle \equiv \{\sigma\} \quad \text{and} \quad \langle\varepsilon\rangle \equiv \{\varepsilon\} \tag{9}$$

which are well known to be directly calculable in terms of the surface tractions and displacements. On the other hand,

$$\langle L\rangle \neq \{L\} \quad \text{and} \quad \langle M\rangle \neq \{M\}$$

when the material is elastically heterogeneous. Mean-value formulae for the macro-

scopic moduli and compliances can, however, be obtained as follows.

Let τ and η be the microscopic stress and strain in an imagined all-elastic state. The existence of a microscopic potential energy is assumed, so that L and M have diagonal symmetry when their components are arranged as 9×9 matrices in the customary way; see [8] for example. Let A and B be the fourth-order "concentration factor" tensors such that

$$\eta = A\,\langle\eta\rangle, \qquad \tau = B\,\langle\tau\rangle \tag{10}$$

for a considered type of heterogeneity. Tensors A and B were introduced and used extensively by Hill [2, 4]; B was also mentioned in passing by Mandel. A and B are, in principle, uniquely determinate functions of position throughout the representative element (whose shape may without loss of generality be supposed a cube). To be precise we should exclude a negligible surface layer which is affected by microscopic differences in any macro-uniform boundary data producing the same average stress and strain. Now

$$\tau = L\eta, \ \langle\tau\rangle = \{L\eta\} = \langle L\rangle\,\langle\eta\rangle; \ \eta = M\tau, \langle\eta\rangle = \{M\tau\} = \langle M\rangle\langle\tau\rangle \tag{11}$$

and so

$$\langle L\rangle = \{LA\}, \ \langle M\rangle = \{MB\} \tag{12}$$

are the required formulae for the macroscopic moduli and compliances.

As remarked in [4], further consequences of (10) and (11) are the equivalent connexions

$$A\langle M\rangle = MB, \qquad B\langle L\rangle = LA \tag{13}$$

between the concentration-factor functions. (For example, the second relation follows by expressing τ in the alternative ways $LA\langle\eta\rangle$ and $B\langle L\rangle\langle\eta\rangle$). The 9×9 matrix representations of $\langle L\rangle$ and $\langle M\rangle$ have diagonal symmetry [2, 4], since the differential form $\langle\tau\rangle\, d\langle\eta\rangle$ is integrable in view of (1) and the assumed integrability of $\tau d\eta$ everywhere. On the other hand, the matrix representations of A and B are not symmetric in general, so their transposes will be distinguished by the notation A' and B'.

Consider, next, the application of theorem (1) to the scalar products $\sigma\eta$ and $\tau\varepsilon$. With the help of (10) there follows

$$\{\sigma A\}\,\langle\eta\rangle = \langle\sigma\rangle\,\langle\eta\rangle, \qquad \{\varepsilon B\}\,\langle\tau\rangle = \langle\varepsilon\rangle\,\langle\tau\rangle$$

for arbitrary $\langle\eta\rangle$ and $\langle\tau\rangle$ respectively. Thus

$$\{A'\sigma\} = \langle\sigma\rangle, \qquad \{B'\varepsilon\} = \langle\varepsilon\rangle \tag{14}$$

for any self-equilibrated field of stress and any kinematically compatible field of strain, provided each can be generated by macro-uniform boundary data. Still considering the same scalar products, we can transform them differently to extract further information:

$$\{\varepsilon LA\}\,\langle\eta\rangle = \{\varepsilon L\eta\} = \langle\varepsilon\rangle\,\langle L\rangle\,\langle\eta\rangle, \ \{\sigma MB\}\,\langle\tau\rangle = \{\sigma M\tau\} = \langle\sigma\rangle\,\langle M\rangle\,\langle\tau\rangle \tag{15}$$

for arbitrary $\langle\eta\rangle$ and $\langle\tau\rangle$. Hence

$$\{A'L\varepsilon\} = \langle L\rangle\,\langle\varepsilon\rangle, \ \{B'M\sigma\} = \langle M\rangle\,\langle\sigma\rangle \tag{16}$$

Alternatively, these can be derived by combining the transpose of (13) with (14). Finally, by subtracting (16) from (14):

$$\{A'(\sigma - L\varepsilon)\} = \langle\sigma\rangle - \langle L\rangle\,\langle\varepsilon\rangle, \ \{B'(\varepsilon - M\sigma)\} = \langle\varepsilon\rangle - \langle M\rangle\,\langle\sigma\rangle \tag{17}$$

when the representative element is partly plastic. These equations are essentially equivalent, since each can be derived from the other by means of (13).

In all the preceding formulae it is of course permissible to replace stress and strain by their differentials or by their rates of change. By reference to the quantities (2) and (3), and to their duals (5) and (6), we see that (17) expresses their macroscopic values in terms of weighted averages of their microscopic distributions. It is remarkable that, regardless of the character and extent of the internal plastic flow, the appropriate weighting functions A' and B' are calculable by elastic theory alone.

Some authors, for example Lin and Ito [9] and Havner [10], have defined the macroscopic plastic strain as a simple unweighted average of its microscopic distribution. But $\{M\sigma\}$ is not equal to $\langle M\rangle\langle\sigma\rangle$ in general, unless the distribution of M is homogeneous or σ is an elastic field. Consequently, such a definition has in general no operational significance. That is, it does not correspond to the actual experimental procedure of unloading a representative element by suitably altering the tractions applied to its surface.

3. Macroscopic scalars. We turn now to quantities having the character of energy or work, either actual or virtual.

To begin with, consider the positive-definite quadratic forms

$$V = {}^1/_2(\varepsilon - \eta)\, L\, (\varepsilon - \eta), \quad W = {}^1/_2\ (\sigma - \tau)\, M\, (\sigma - \tau) \tag{18}$$

where η and τ are elastic fields, as in (10), but now such that $\langle\eta\rangle = \langle\varepsilon\rangle$ and $\langle\tau\rangle = \langle\sigma\rangle$. The global averages, $\{V\}$ and $\{W\}$, can be interpreted as the residual elastic energies in the representative element after removal of the macroscopic strain $\langle\varepsilon\rangle$ and macroscopic stress $\langle\sigma\rangle$ respectively (provided that this removal is imagined to be a perfectly elastic process). We recall the elementary formulae

$$2\{V\} = \{\varepsilon L\varepsilon\} - \langle\varepsilon\rangle\langle L\rangle\langle\varepsilon\rangle, \quad 2\{W\} = \{\sigma M\sigma\} - \langle\sigma\rangle\langle M\rangle\langle\sigma\rangle \tag{19}$$

whose analogues for elastic continua under general loading are well known (more especially in connexion with the classical extremum principles). To prove (19) note that $L\eta$ is a self-equilibrated field of stress while $M\tau$ is a kinematically compatible field of strain. Then (18) can be reduced to

$$2\{V\} = \{(\varepsilon - \eta)\, L\varepsilon\}, \qquad 2\{W\} = \{(\sigma - \tau)\, M\sigma\}$$

with the help of (1), and then to (19) by relations similar to (11).

Taking the differentials of expressions (19) we obtain

$$d\{V\} = \{\varepsilon L d\varepsilon\} - \langle\varepsilon\rangle\langle L\rangle d\langle\varepsilon\rangle, \quad d\{W\} = \{\sigma M d\sigma\} - \langle\sigma\rangle\langle M\rangle d\langle\sigma\rangle \tag{20}$$

assuming that L and M do not vary. Re-written, again with the help of (1), in a form suited to elastoplastic continua, (20) becomes

$$\begin{aligned} \langle\varepsilon\rangle(d\langle\sigma\rangle - \langle L\rangle d\langle\varepsilon\rangle) - \{\varepsilon\,(d\sigma - L d\varepsilon)\} &= d\{V\} \\ \langle\sigma\rangle(d\langle\varepsilon\rangle - \langle M\rangle d\langle\sigma\rangle) - \{\sigma\,(d\varepsilon - M d\sigma)\} &= d\{W\} \end{aligned} \tag{21}$$

The macroscopic and microscopic measures of the plastic work in an increment of strain are thereby explicitly distinguished [1].

More precisely, when reckoned from the virgin elastic state of a representative element, the total macroscopic plastic work always exceeds the total microscopic plastic

work. This excess is equal to the final value of the residual elastic energy $\{W\}$. However, even if the microscopic plastic work increases monotonically (supposing a perpetually positive rate of energy dissipation in the micro-constituants), the macroscopic plastic work may nevertheless fluctuate and temporarily decrease. For example, as remarked in [2], the macroscopic plastic work-rate in a uniaxial tension test is negative during that part of a hysteresis loop where reversed plastic flow occurs before the tensile load has been completely removed. This type of hysteresis loop is, of course, often observed in tests on polycrystalline metals.

Next we compare virtual work-rates of the microscopic and macroscopic levels. In analogy to (15) we have

$$\{\eta^* L d\varepsilon\} = \langle \eta^* \rangle \langle L \rangle \, d\langle \varepsilon \rangle, \qquad \{\tau^* M d\sigma\} = \langle \tau^* \rangle \langle M \rangle d\langle \sigma \rangle$$

where τ^* and η^* are any elastic fields of stress and strain respectively. Immediate consequences are

$$\begin{aligned} \{\eta^* (d\sigma - L d\varepsilon)\} &= \langle \eta^* \rangle (d\langle \sigma \rangle - \langle L \rangle \, d\langle \varepsilon \rangle) \\ \{\tau^* (d\varepsilon - M d\sigma)\} &= \langle \tau^* \rangle (d\langle \varepsilon \rangle - \langle M \rangle \, d\langle \sigma \rangle) \end{aligned} \tag{22}$$

Apart from sign these equations are identical on account of (11). Various versions of them can be found in [1, 2, 3].

Now $\eta^* = A\langle \eta^* \rangle$ and $\tau^* = B\langle \tau^* \rangle$, as in (10). Since these fields are arbitrary, the differential analogues of (17) can be recovered from (22). Suppose, on the other hand, that the field τ^* is regarded as the actual (infinitesimal) change in stress during (partial) elastic unloading from the state σ. Then (22) becomes a statement that micro-normality implies macro-normality (both sides of the equation being non-positive). This fundamental result was derived independently by Mandel and Hill.

A general comparison of the virtual plastic work-rate at the microscopic and macroscopic levels was also undertaken by Mandel. His analysis may be re-phrased as follows. Consider the difference

$$\langle \sigma^* \rangle \, (d\langle \varepsilon \rangle - M d\langle \sigma \rangle) - \{\sigma^* (d\varepsilon - M d\sigma)\} \tag{23}$$

where σ^* is any self-equilibrated field of stress. By (1) this difference reduces to

$$\{\sigma^* M d\sigma\} - \langle \sigma^* \rangle \langle M \rangle d\langle \sigma \rangle$$

which can be converted either

$$\{(\sigma^* - \tau^*) \, M d\sigma\} \quad \text{or} \quad \{\sigma^* M (d\sigma - d\tau)\}$$

where τ^* and $d\tau$ are elastic fields of stress under $\langle \sigma^* \rangle$ and $d\langle \sigma \rangle$ respectively. From the foregoing quantities we subtract respectively the expressions

$$\{(\sigma^* - \tau^*) M d\tau\} \quad \text{and} \quad \{\tau^* M (d\sigma - d\tau)\}$$

which are both in fact zero. (In each product the bracketed factor is a self-equilibrated field of stress with a null average, while the other factor is a kinematically compatible field of strain). The result by either route is

$$\{(\sigma^* - \tau^*) M (d\sigma - d\tau)\} \tag{24}$$

The reduction of (23) to (24) is Mandel's formula. Its dual may be proved similarly, and states that the difference

$$\langle \varepsilon^* \rangle \, (d\langle \sigma \rangle - \langle L \rangle \, d\langle \varepsilon \rangle) - \{\varepsilon^* (d\sigma - L d\varepsilon)\} \tag{25}$$

is equal to

$$\{(\varepsilon^* - \eta^*)L(d\varepsilon - d\eta)\} \tag{26}$$

By specializing Mandel's general formula and its dual in appropriate ways all the earlier results can be regained. Thus, the choice $\sigma^* = \sigma$ (with $\tau^* = \tau$ necessarily) in (23) and (24) reproduces the second equation (21); likewise $\varepsilon^* = \varepsilon$ (with $\eta^* = \eta$ necessarily) in (25) and (26) reproduces the first equation (21). On the other hand, the choice $\sigma^* = \tau^*$ returns to (22); naturally $\varepsilon^* = \eta^*$ has the same effect.

By taking σ^* proportional to $d\sigma$ (and τ^* proportional to $d\tau$) Mandel also obtained a new result:

$$\begin{aligned} d\langle\sigma\rangle\,(d\langle\varepsilon\rangle - \langle M\rangle d\langle\sigma\rangle) - \{d\sigma(d\varepsilon - Md\sigma)\} \\ = \{(d\sigma - d\tau)M(d\sigma - d\tau)\} \end{aligned} \tag{27}$$

Its dual is

$$\begin{aligned} d\langle\varepsilon\rangle\,(d\langle\sigma\rangle - \langle L\rangle d\langle\varepsilon\rangle) - \{d\varepsilon(d\sigma - Ld\varepsilon)\} \\ = \{(d\varepsilon - d\eta)\,L\,(d\varepsilon - d\eta)\} \end{aligned} \tag{28}$$

Equations (27) and (28) could of course be derived directly from the differential analogues of (19).

Now $d\sigma \not\equiv d\tau$ and $d\varepsilon \not\equiv d\eta$, unless the deformation is purely elastic. Consequently, from (27) and (28),

$$\begin{aligned} d\langle\sigma\rangle\,(d\langle\varepsilon\rangle - \langle M\rangle d\langle\sigma\rangle) &> \{d\sigma(d\varepsilon - Md\sigma)\} \\ d\langle\varepsilon\rangle\,(d\langle\sigma\rangle - \langle L\rangle d\langle\varepsilon\rangle) &> \{d\varepsilon(d\sigma - Ld\varepsilon)\} \end{aligned} \tag{29}$$

whenever incremental plastic deformations occur. An independent direct proof was given in [2]. We can infer immediately that, if

$$d\sigma\,(d\varepsilon - Md\sigma) \geqslant 0 \tag{30}$$

for every micro-constituent, then

$$d\langle\sigma\rangle\,(d\langle\varepsilon\rangle - \langle M\rangle d\langle\sigma\rangle) > 0 \tag{31}$$

As emphasized in [2], this last inequality applies even when the equality in (30) holds throughout the representative element. That is to say (assuming the normality rule), an apparent macroscopic hardening would be observed, because of the heterogeneity, even when the micro-constituents did not harden.

By the differential analogues of (8) for the microscopic and macroscopic variables it can be concluded that (30) implies not only

$$d\varepsilon(d\sigma - Ld\varepsilon) \leqslant 0 \tag{32}$$

but also

$$d\langle\varepsilon\rangle\,(d\langle\sigma\rangle - \langle L\rangle\,d\langle\varepsilon\rangle) \leqslant 0 \tag{33}$$

On the other hand, nothing can be decided from the second of (29) as to whether (32) by itself implies (33). Indeed (32) merely leads to

$$d\langle\sigma\rangle\,d\langle\varepsilon\rangle \leqslant \{d\varepsilon\ \ (Ld\varepsilon)\}$$

which can be carried no farther because

$$\{d\varepsilon(Ld\varepsilon)\} \geqslant d\langle\varepsilon\rangle\ (\langle L\rangle\,d\langle\varepsilon\rangle)$$

by analogy with the first of (19).

Now (32) and (33) would be consequences of asserting Ilyushin's postulate at the microscopic and macroscopic levels respectively (applied, in particular, to an infinitesimal

cycle of strain starting from a plastic state of stress). On the other hand, Ilyushin's postulate does not suffice for (30) and (31) (though Drucker's does, as is well known). These facts prompt an interesting basic question: namely, whether any relation exist between the microscopic and macroscopic forms of Ilyushin's postulate and, in particular, whether the two forms are even always mutually compatible. Needless to say, this question does not admit an easy answer, since a cycle of macroscopic strain is by no means necessarily accompanied by cycles of strain in all the micro-constituents.

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